Container Sieve Hierarchies, Colossally Abundant Numbers, and Spectral Filtering for the Riemann Hypothesis

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Abstract

We introduce a tiered framework of integer sieves—standard primes, power primes, and factorial primes—formulated through container state algebra. By analyzing the structural exponents of colossally abundant numbers (CANs) within these container spaces, we prove that all CANs are invariant under both power-prime and factorial-prime sieves. Furthermore, we investigate the "factorial composite resonance chamber" (squares and higher factorial powers) and demonstrate how its divisor sum error terms serve as a spectral filter for the non-trivial zeros of the Riemann zeta function, governing the stability of Robin's inequality.

1 Introduction and Container Sieve Definitions

The classical Sieve of Eratosthenes operates by eliminating linear multiples $(n, 2n, 3n, \dots)$. By generalizing this process to eliminate higher powers and factorial indices, we construct a hierarchy of container sieves. Let an integer $n > 1$ possess the canonical prime factorization $n = \prod_{i=1}^{r} p_i^{a_i}$.

Definition 1 (Standard Primes)**.** Standard primes are integers $n > 1$ whose exponent vector under the linear multiple sieve contains a single component equal to 1 with all other factors absent, yielding the traditional set of prime numbers $\mathbb{P}$.

Definition 2 (Power Primes (OEIS A007916))**.** An integer $n > 1$ is termed a power prime (or non-perfect power) if it is not of the form m^k for any integer $k \geq 2$. Equivalently, the greatest common divisor of its prime exponents is unity:

$$\gcd(a_1, a_2, \dots, a_r) = 1 \tag{1}$$

Definition 3 (Factorial Primes)**.** An integer $n > 1$ is termed a factorial prime if it cannot be expressed in the form $m^{k!}$ for any integer $m \geq 2$ and $k \geq 1$. Equivalently, no factorial index $k! \geq 2$ divides all prime exponents simultaneously:

$$\forall k \geq 1, \quad k! \nmid \gcd(a_1, a_2, \dots, a_r) \tag{2}$$

2 Analogs and Inclusion Hierarchies

The three prime definitions form a strict structural inclusion hierarchy based on the elimination of progressive exponent families:

$$\mathbb{P} \subsetneq \text{Power Primes} \subsetneq \text{Factorial Primes} \subsetneq \mathbb{N} \tag{3}$$

Proposition 1 (Factorial Collapse)**.** *Since every factorial index $k!$ for $k \geq 2$ is an even multiple of 2 ($2! = 2$ divides $k!$), the factorial-prime sieve eliminates precisely the integer squares and higher factorial powers. Consequently, the set of factorial composites consists strictly of numbers of the form $m^{k!} = (m^{k!/2})^2$, collapsing structurally to squares.*

3 Colossally Abundant Numbers and Container Invariance

Colossally abundant numbers (CANs) maximize the growth metric $\frac{\sigma(N)}{N^{1+\varepsilon}}$ for some $\varepsilon > 0$ and serve as the critical testbed for Robin's inequality.

Lemma 1 (Tail Exponent Unit Bound [2])**.** *Let $N = \prod_{i=1}^{m} p_i^{a_i}$ be a colossally abundant number. The prime exponents are governed by $a_i = \lfloor (p_i^{\varepsilon} - 1)^{-1} \rfloor$. For the largest prime factor p_m, the exponent is strictly $a_m = 1$.*

Theorem 1 (Inevitability in Sieve Spaces)**.** *Every colossally abundant number beyond the trivial range is both a power prime and a factorial prime.*

Proof. By Lemma 1, every CAN possesses a largest prime factor with exponent $a_m = 1$. Evaluating the greatest common divisor yields $\gcd(a_1, \ldots, a_{m-1}, 1) = 1$, satisfying the power-prime condition. Furthermore, since 1 is not divisible by any factorial $k! \geq 2$, the CAN satisfies the factorial-prime condition. Thus, CANs are entirely immune to both sieves. □

4 Robin's Inequality and the Riemann Hypothesis

Robin's inequality states that for all $n > 5040$:

$$\sigma(n) < e^{\gamma} n \log\log n \tag{4}$$

Robin (1984) proved this inequality holds universally if and only if the Riemann Hypothesis is true [1]. Because any minimal counterexample must be a colossally abundant number, and all CANs reside within the factorial-prime and square-free container space, structural violations are strictly governed by the container error terms.

Theorem 2 (Factorial Composite Resonance Chamber)**.** *The divisor summatory function restricted to factorial composites (squares),* $S_{FC}(x) = \sum_{m \leq \sqrt{x}} \sigma(m^2)$*, generates error terms controlled by the Dirichlet series ratio* $\frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}$*. These error fluctuations act as a spectral filter for the non-trivial zeros of* $\zeta(s)$*, determining whether CAN divisor sums can breach the asymptotic ceiling of Robin's inequality.*

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References

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